



The Open
University

MST124

Essential mathematics 1

Exercise Booklet 3

1 Functions and their graphs

Exercise 1

Let X be the set of letters in the word 'question'. Specify X by listing its elements in curly brackets.

Exercise 2

Let $A = \{2, 4, 8, 16\}$, let B be the set of all positive multiples of 3 and let C be the set of all positive factors of 36. Find the elements of each of the following sets.

- (a) C (b) $A \cup C$ (c) $B \cap C$
 (d) $A \cap B$ (e) $(A \cup B) \cap C$

Exercise 3

For each of the following inequalities and double inequalities, write the interval described in interval notation, and state whether it is open, closed or half-open.

- (a) $-3 \leq x < -2$ (b) $-3 \leq x$
 (c) $-3 < x < 2$ (d) $x < 2$

Exercise 4

In each part below, draw a number line diagram similar to those in Subsection 1.1 of Unit 3 to illustrate the set described.

- (a) $2 \leq x \leq 4$ (b) $\{3, 4, 5\}$

Exercise 5

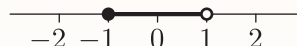
Which of the following intervals contain the number -2 ?

- $(-5, 0)$, $[-3, -2)$, $(-1, 3)$,
 $(-3, -2]$, $(-2, \infty)$

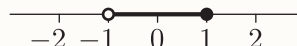
Exercise 6

Put items (a)–(h) into four pairs, where the two items in each pair represent the same interval.

- (a) $[-1, 1]$ (b) $-1 \leq x < 1$
 (c) $(-1, 1]$ (d) $-1 \leq x \leq 1$
 (e) $(-1, 1)$ (f) $-1 < x < 1$
 (g)



(h)



Exercise 7

For each of the following sets, state whether it is an interval.

- (a) $[-10, 1) \cup (-1, 10]$
 (b) $\mathbb{R} \cap [-1, 1]$
 (c) $[-10, 0) \cup (0, 10]$

Exercise 8

For each of parts (a) and (b), write down a reason to explain why it does *not* specify a function.

- (a) $f(x) = \pm\sqrt{x-1} \quad (x \geq 1)$
 (b) $f(x) = \sqrt{x-2} \quad (1 \leq x \leq 6)$

Exercise 9

For each of the following functions, write down its domain using interval notation.

- (a) $f(x) = x^2 + 3 \quad (x \geq 0)$
 (b) $f(x) = (\sqrt{x-1})^3$
 (c) $f(x) = \frac{x}{x+1} + \frac{3}{\sqrt{x+2}}$

Exercise 10

For each of the following functions, find $f(-2)$.

$$(a) f(x) = \begin{cases} x^3 + 3 & \text{for } x < -1 \\ x^3 - 3 & \text{for } x \geq -1 \end{cases}$$

$$(b) f(x) = \begin{cases} x^3 + 3 & \text{for } x < -2 \\ x^3 - 3 & \text{for } x \geq -2 \end{cases}$$

Exercise 11

Draw the graph of the following function.

$$f(x) = \begin{cases} -2x - 1 & \text{for } x \in [-5, -1) \\ x + 1 & \text{for } x \in [-1, 1] \\ 2x - 3 & \text{for } x \in (1, 5] \end{cases}$$

Exercise 12

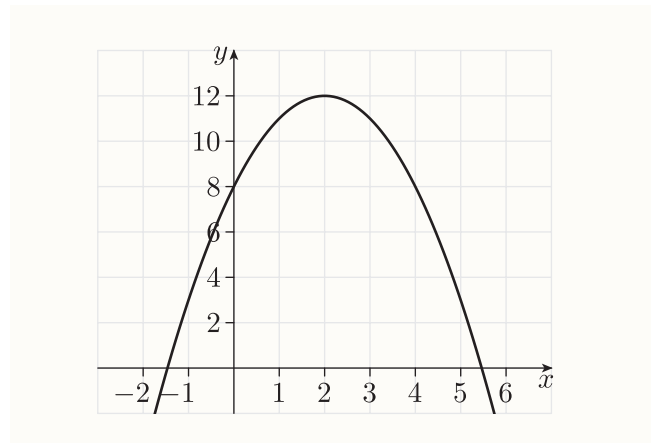
A circular pond of radius r metres is constructed at the centre of a 10 metre by 15 metre rectangular plot, and the remainder of the plot is sown with grass seed to give a lawn of area A square metres.

- Write down the largest closed interval containing values of r that make sense for this problem.
- Express A in terms of r .
- Find the radius of the pond if the area of the lawn is two-thirds of the area of the whole plot.

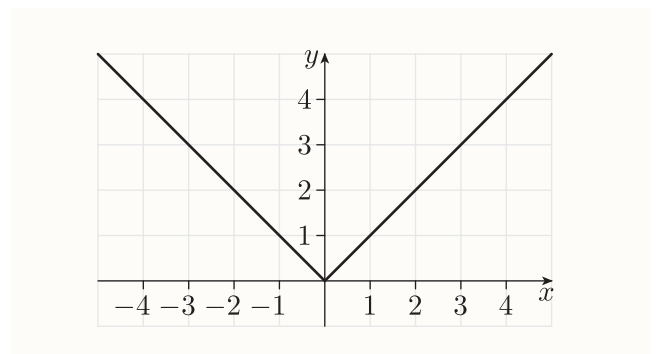
Exercise 13

For each of the following graphs, write down an interval on which the corresponding function is decreasing. Use interval notation.

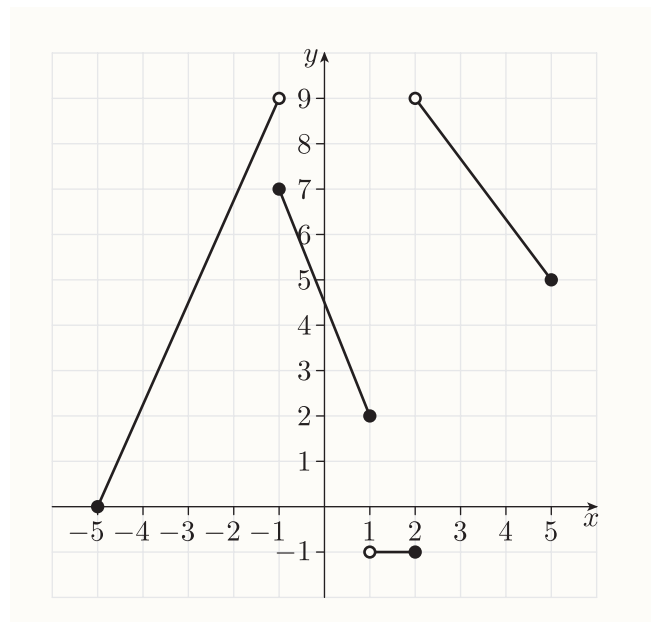
(a)



(b)



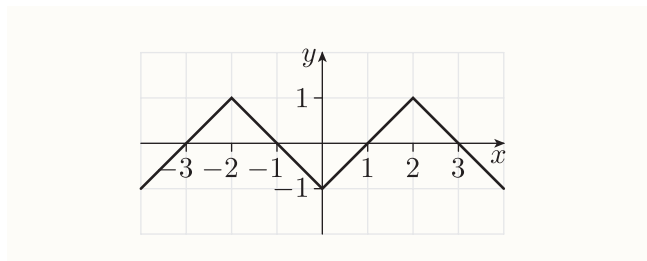
(c)



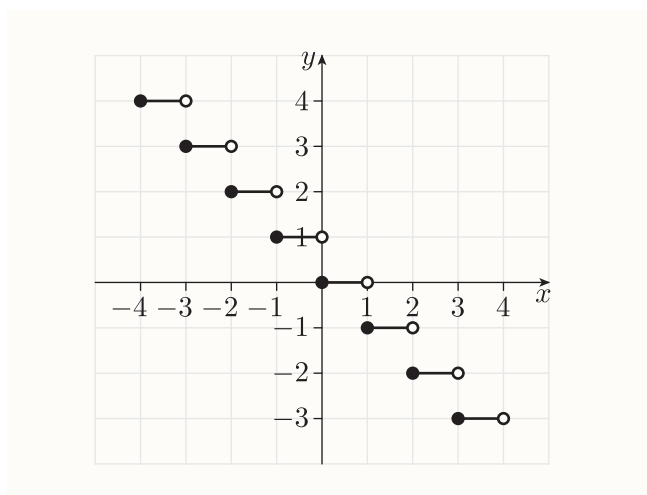
Exercise 14

In each of parts (a) and (b), write down the image set of the function whose graph is shown.

(a)



(b)



Exercise 15

For each of the following quadratic functions, find the coordinates of the vertex of the parabola that is its graph, and hence write down its image set.

(a) $f(x) = x^2 - 16x + 40$,

(b) $f(x) = -4x^2 + 4x + 2$.

Hint: in each part start by completing the square.

Exercise 16

For each of the following quadratic functions, write down an interval on which it is increasing, and an interval on which it is decreasing.

Hint: the working in the solution to the previous exercise should be helpful.

(a) $f(x) = x^2 - 16x + 40$

(b) $f(x) = -4x^2 + 4x + 2$

Exercise 17

Write down the degree of each of the following polynomial functions.

(a) $f(x) = -4x^5 + 17x^3$

(b) $f(t) = 3t - t^2 - 3t^3$

(c) $f(x) = 3 - x^k$, where k is a natural number

(d) $f(t) = \ln 2$

Exercise 18

How are the following two rational functions related to each other?

$$f(x) = \frac{2 - x^2}{3x^3 + 4} \quad \text{and} \quad g(x) = \frac{3x^3 + 4}{2 - x^2}$$

Exercise 19

Sketch the graph of the function

$$f(x) = \begin{cases} \frac{x^2 - 10}{2} & \text{for } x < -2 \\ x + 1 & \text{for } x \geq -2. \end{cases}$$

2 New functions from old functions

Exercise 20

Find the coordinates of the corners of the graphs of the following functions.

- (a) $f(x) = |x| - 2$
 - (b) $f(x) = |x - 2|$
 - (c) $f(x) = |x + 3| - 4$
-

Exercise 21

For each of the following functions, describe how you could obtain its graph by applying scalings and translations to the graph of the function $f(x) = |x|$, and sketch the graph of the function.

- (a) $f(x) = \frac{1}{4}|x - 2|$
 - (b) $f(x) = -2|x + 1|$
 - (c) $f(x) = -2(|x - 1| - 3)$
-

Exercise 22

For each of the following quadratic functions, describe how you could obtain its graph by applying scalings and translations to the graph of the function $f(x) = x^2$. In each case also find the x -intercepts (if any) and y -intercepts, and sketch the graph.

- (a) $f(x) = 3x^2 + 18x + 15$
 - (b) $f(x) = -3x^2 + 2x - \frac{7}{3}$
-

3 More new functions from old functions

Exercise 23

This question is about the functions

$$f(x) = (x + 3)^3 + 8 \quad \text{and} \quad g(x) = |x| + 3.$$

Write down the rule and domain of each of the following functions.

- (a) The sum of f and g .
 - (b) The difference of f and g , with g subtracted from f .
 - (c) The quotient of f and g , with f as the numerator.
 - (d) The other quotient of f and g .
-

Exercise 24

This question is about the functions

$$f(x) = x^3, \quad g(x) = \sqrt{x}, \quad h(x) = x - 1.$$

Find the rules of the following composite functions.

- (a) $g \circ h$ (b) $h \circ g$ (c) $f \circ g$
 - (d) $g \circ f$ (e) $f \circ g \circ h$
-

Exercise 25

This question is about the functions

$$f(x) = |x|, \quad g(x) = x - 2, \quad h(x) = \frac{1}{x}.$$

The following functions are all formed by composing f or g or h with f or g or h . Write each of them using composite function notation.

- (a) $k(x) = |x| - 2$ (b) $k(x) = |x - 2|$
 - (c) $k(x) = x$
-

Exercise 26

Which of the following functions are one-to-one? Justify your answers.

- (a) $f(x) = x^3 + 1$
 (b) $f(x) = x(x - 2) + 1$
 (c) $f(x) = x^3 - x$
 (d) $f(x) = ax^2$, where $a \neq 0$

Exercise 27

Find the inverse of each of the following functions.

- (a) $f(x) = \frac{(1-x)}{3}$ (b) $f(x) = 5 - \frac{1}{x}$
 (c) $f(x) = \frac{1}{5-x}$

Exercise 28

For each of the following functions f , find the inverse function f^{-1} , and sketch the graph of f^{-1} .

- (a) $f(x) = 2 - 3x$
 (b) $f(x) = 2x + 2$ ($x \in (-1, 2)$)

Exercise 29

Consider the function $f(x) = x^2 - 4x + 7$.

- (a) Complete the square in the rule of f .
 (b) Sketch the graph of f .
 (c) Using your sketch, specify a one-to-one function g that is a restriction of f and has the same image set as f .
 (d) Find the inverse function g^{-1} of g .

4 Exponentials and logarithms

Exercise 30

Pair each exponential function below with its approximately equivalent form. (Try to do this without using a calculator.)

- (a) $f(x) = 1000^x$ (b) $f(x) = 0.5^x$
 (c) $f(x) = e^{6.907755x}$
 (d) $f(x) = 2.718282^x$
 (e) $f(x) = e^{-0.693147x}$ (f) $f(x) = e^x$

Exercise 31

Find the exact value of each of the following expressions, without using your calculator.

- (a) $\log_3 1$ (b) $\log_5(\frac{1}{25})$
 (c) $\ln(e^{3.1})$ (d) $\log_{10}(\sqrt{1000})$
 (e) $e^{\ln 4}$ (f) $2^{3\log_2 5}$

Exercise 32

For each of the following equations, find the value of x that satisfies it, if there is such a value.

- (a) $3 = \log_5 x$ (b) $x = \log_{\sqrt{2}}(-2)$
 (c) $\frac{1}{2} = \log_x 9$ (d) $x = \log_{-1} \frac{1}{2}$

Exercise 33

Find the exact value of the expression

$$e^{\ln 2} + \log_5 \left(\frac{1}{125} \right) + \ln(4e) - \ln 4,$$

without using your calculator.

Exercise 34

For each of the following functions, describe how you could obtain its graph by applying scalings and translations to the graph of the function $f(x) = e^x$, and sketch the graph of the function.

(a) $g(x) = e^{x-1} + 1$ (b) $h(x) = e^{x/2}$

Exercise 35

Solve the following exponential equations.

(a) $3^x = 70$

(b) $3^x = 81$

Exercise 36

Use the logarithm laws to show that each of the following equations is true for all values of its variables.

(a) $\ln\left(\frac{1}{2}xy^3\right) = \ln x + 3\ln y - \ln 2$

(b) $\ln\left(\frac{3x^2 - 12}{e^{2x}}\right) = \ln 3 + \ln(x - 2) + \ln(x + 2) - 2x$

Exercise 37

A beefburger is infected with salmonella bacteria, which begin to multiply after 2 hours. Tests 6 hours and 8 hours after the time of the initial infection show that the beefburger contains about 500 and 5000 bacteria per gram, respectively. Assume that the number of bacteria in the beefburger can be modelled using an exponential growth function f , where $f(t)$ is the number of bacteria per gram after t hours, for $2 \leq t \leq 12$.

- Find the function f , stating the constant in the exponent in its rule to four significant figures.
- What is the expected number of bacteria per gram after 12 hours?
- By what factor is the size of the population of bacteria multiplied every hour?

5 Inequalities

Exercise 38

Assuming that $x > 2$, determine whether each of the following inequalities is true or false, or whether it is not possible to say.

(a) $4x > 8$ (b) $x - 2 > 0$

(c) $\frac{x}{2} > 1$ (d) $-x > -2$

(e) $-\frac{x}{2} > -1$ (f) $ax > 2a$, where $a \in \mathbb{R}$

Exercise 39

Solve the following inequalities. Give each solution set in interval notation.

(a) $5x + 2 < 2x + 11$

(b) $5x - 2 \leq 2x - 11$

(c) $\frac{x - 2}{3} > 1 - \frac{x}{2}$

Exercise 40

Solve the following inequalities.

(a) $x^2 - x > 6$

(b) $-2x^2 < 2$

(c) $x^2 \leq 2x - 1$

Exercise 41

Solve the inequality

$$3x + 7 < \frac{20}{3x - 1}$$

by using a table of signs.

Solutions to exercises

Solution to Exercise 1

$$X = \{e, i, n, o, q, s, t, u\}.$$

(You can list the elements in any order. For example, $X = \{q, u, e, s, t, i, o, n\}$ is a correct answer.)

Solution to Exercise 2

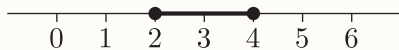
- (a) $C = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$
- (b) $A \cup C = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 36\}$
- (c) $B \cap C = \{3, 6, 9, 12, 18, 36\}$
- (d) $A \cap B = \emptyset$
- (e) $(A \cup B) \cap C = \{2, 3, 4, 6, 9, 12, 18, 36\}$

Solution to Exercise 3

- (a) $[-3, -2)$ is half-open.
- (b) $[-3, \infty)$ is closed.
- (c) $(-3, 2)$ is open.
- (d) $(-\infty, 2)$ is open.

Solution to Exercise 4

(a)



(b)



Solution to Exercise 5

Of the five intervals listed, only $(-5, 0)$ and $(-3, -2]$ contain the number -2 .

Solution to Exercise 6

The pairs are as follows:

- (a) and (d), (b) and (g),
- (c) and (h), (e) and (f).

Solution to Exercise 7

- (a) This set is an interval.
The union of the intervals $[-10, 1)$ and $(-1, 10]$ is the interval $[-10, 10]$.
- (b) This set is an interval.
This intersection of the intervals $\mathbb{R}$ and $[-1, 1]$ is the interval $[-1, 1]$.
- (c) This set is not an interval.
It has a 'gap' at 0.

Solution to Exercise 8

- (a) A function converts each input number into *exactly one* output number. The rule $f(x) = \pm\sqrt{x-1}$ converts each input number x (except 1) into *two* output numbers.
- (b) A function converts *each* input number in its domain into *an* output number. The rule $f(x) = \sqrt{x-2}$ does not convert some input numbers in the set $[1, 6]$ into an output number. For example, $1 \in [1, 6]$ but when $x = 1$ we have $\sqrt{x-2} = \sqrt{-1}$, and -1 does not have a real square root.
(A suitable domain here would be $[2, \infty)$, rather than $[1, 6]$.)

Solution to Exercise 9

- (a) The domain is $[0, \infty)$.
- (b) The expression $\sqrt{x-1}$ is defined for $x-1 \geq 0$, which is equivalent to $x \geq 1$. Therefore the expression $(\sqrt{x-1})^3$ is defined for $x \geq 1$. Hence, by the domain convention, the domain is $[1, \infty)$.
- (c) The expression $x/(x+1)$ is defined for all x in $\mathbb{R}$ except $x = -1$ (since it is not possible to divide by 0).

The expression $\sqrt{x+2}$ is defined for $x+2 \geq 0$, which is equivalent to $x \geq -2$. However the expression $3/\sqrt{x+2}$ is not defined when $x = -2$ (again since it is not possible to divide by 0). So the expression $3/\sqrt{x+2}$ is defined for $x > -2$.

Therefore the domain of the given function is $(-2, \infty)$ with -1 excluded. That is, it is $(-2, -1) \cup (-1, \infty)$.

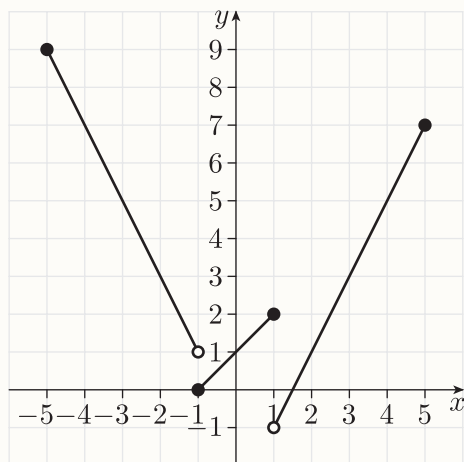
Solution to Exercise 10

- (a) Since
- $-2 < -1$
- ,

$$f(-2) = (-2)^3 + 3 = -5.$$

- (b) Since
- $-2 \geq -2$
- ,

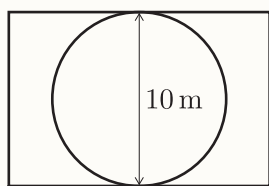
$$f(-2) = (-2)^3 - 3 = -11.$$

Solution to Exercise 11

(You need to use $\bullet$ to show that a point at the end of a piece of the graph is included, and $\circ$ to show that a point is excluded.)

Solution to Exercise 12

- (a) The diameter of the pond, which is $2r$, cannot exceed the smaller of the dimensions of the plot, namely 10 metres, and r cannot be negative. So the largest closed interval of possible values for r is $[0, 5]$.



- (b) The area of the lawn is the area of the rectangular plot minus the area of the pond, so

$$\begin{aligned} A &= 10 \times 15 - \pi r^2 \\ &= 150 - \pi r^2. \end{aligned}$$

- (c) The area of the plot is 150 m^2 , so two-thirds of the area of the plot is 100 m^2 . Hence r must satisfy the equation

$$150 - \pi r^2 = 100;$$

that is,

$$r^2 = \frac{50}{\pi}.$$

The solutions of this equation are

$$r = \pm \sqrt{\frac{50}{\pi}};$$

that is,

$$r = \pm 3.99 \text{ (to 3 s.f.)}.$$

Of the two solutions, only 3.99 lies in the interval $[0, 5]$.

Thus the required radius is 3.99 metres (to 3 s.f.).

Solution to Exercise 13

- (a) The function is decreasing on the interval $[2, \infty)$, and also on every interval that is a subset of this interval.
- (b) The function is decreasing on the interval $(-\infty, 0]$, and also on every interval that is a subset of this interval.
- (c) The function is decreasing on each of the intervals $[-1, 1]$ and $(2, 5]$, and also on every interval that is a subset of one of these intervals.

Solution to Exercise 14

- (a) The image set is $[-1, 1]$.
- (b) The image set is $\{-3, -2, -1, 0, 1, 2, 3, 4\}$.

Solution to Exercise 15

- (a) Completing the square gives

$$\begin{aligned} f(x) &= x^2 - 16x + 40 \\ &= (x - 8)^2 - 64 + 40 \\ &= (x - 8)^2 - 24. \end{aligned}$$

Since $(x - 8)^2$ is a square, it is always non-negative. Therefore the smallest value that $f(x)$ can take is -24 , and this occurs when $x = 8$. So the coordinates of the vertex are $(8, -24)$.

Since the coefficient of x^2 is positive, the parabola is u-shaped.

Hence the image set is $[-24, \infty)$.

(b) Completing the square gives:

$$\begin{aligned} f(x) &= -4x^2 + 4x + 2 \\ &= -4(x^2 - x) + 2 \\ &= -4\left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right) + 2 \\ &= -4\left(x - \frac{1}{2}\right)^2 + 1 + 2 \\ &= -4\left(x - \frac{1}{2}\right)^2 + 3. \end{aligned}$$

Since $\left(x - \frac{1}{2}\right)^2$ is a square, it is always non-negative. Therefore the largest value that $f(x)$ can take is 3, and this occurs when $x = \frac{1}{2}$. So the coordinates of the vertex are $\left(\frac{1}{2}, 3\right)$.

Since the coefficient of x^2 is negative, the parabola is n-shaped. Hence the image set is $(-\infty, 3]$.

Solution to Exercise 16

(a) From the previous question, we know that the graph of $f(x) = x^2 - 16x + 40$ is a u-shaped parabola whose vertex has x -coordinate 8. Therefore this function is

- increasing on $[8, \infty)$ (and on every interval that is a subset of this interval)
- decreasing on $(-\infty, 8]$ (and on every interval that is a subset of this interval).

(b) From the previous question we know that the graph of $f(x) = -4x^2 + 4x + 2$ is an n-shaped parabola whose vertex has x -coordinate $\frac{1}{2}$. Therefore this function is

- increasing on $(-\infty, \frac{1}{2}]$ (and on every interval that is a subset of this interval)
- decreasing on $[\frac{1}{2}, \infty)$ (and on every interval that is a subset of this interval).

Solution to Exercise 17

The degrees of the polynomial functions are as follows.

(a) 5 (b) 3 (c) k (d) 0

(In part (d), the degree is 0 because $\ln 2$ is a constant.)

Solution to Exercise 18

The functions are the reciprocals of each other. In other words,

$$g(x) = \frac{1}{f(x)} \quad \left(\text{and so } f(x) = \frac{1}{g(x)}\right).$$

Solution to Exercise 19

When $x < -2$,

$$f(x) = \frac{x^2 - 10}{2} = \frac{1}{2}x^2 - 5,$$

so the part of the graph for these x -values is part of a u-shaped parabola with vertex $(0, -5)$.

When $x = -2$,

$$\frac{1}{2}x^2 - 5 = \frac{1}{2}(-2)^2 - 5 = -3,$$

so the graph has an endpoint at $(-2, -3)$, which is excluded.

The x -intercept with $x < -2$ is given by

$$\frac{1}{2}x^2 - 5 = 0$$

so it is

$$x = -\sqrt{10} = -3.16 \text{ (to 3 s.f.)}.$$

When $x \geq -2$,

$$f(x) = x + 1,$$

so the part of the graph for these x -values is part of a straight line.

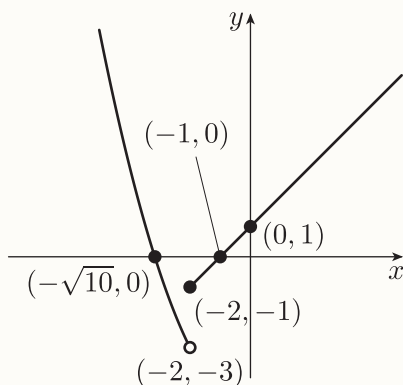
When $x = -2$,

$$x + 1 = -2 + 1 = -1,$$

so the graph has an endpoint at $(-2, -1)$, which is included.

The x -intercept with $x \geq -2$ is given by $x + 1 = 0$, so it is -1 . The y -intercept with $x \geq -2$ is $0 + 1 = 1$.

So the graph is as shown overleaf.



Solution to Exercise 20

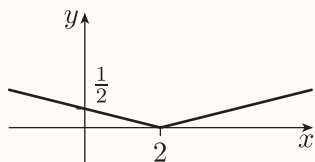
- (a) The graph of $y = |x| - 2$ is obtained from the graph of $y = |x|$ by translating it down by 2 units. Therefore the corner is at $(0, -2)$.
- (b) The graph of $y = |x - 2|$ is obtained from the graph of $y = |x|$ by translating it to the right by 2 units. Therefore the corner is at $(2, 0)$.
- (c) The graph of $y = |x + 3| - 4$ is obtained from the graph of $y = |x|$ by translating it down by 4 units and also translating it to the left by 3 units. Therefore the corner is at $(-3, -4)$.

Solution to Exercise 21

- (a) The graph of the equation $y = \frac{1}{4}|x - 2|$ can be obtained by starting with the graph of $y = |x|$, scaling it vertically by the factor $\frac{1}{4}$, and then translating it to the right by 2 units.

(These transformations can be carried out in either order.)

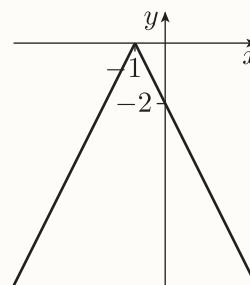
The y -intercept is $f(0) = \frac{1}{4}|0 - 2| = \frac{1}{2}$.



- (b) The graph of the equation $y = -2|x + 1|$ can be obtained by starting with the graph of $y = |x|$, scaling it vertically by the factor -2 , and then translating it to the left by 1 unit.

(These transformations can be carried out in either order.)

The y -intercept is $f(0) = -2|0 + 1| = -2$.



- (c) The graph of the equation $y = -2(|x - 1| - 3)$ can be obtained by starting with the graph of $y = |x|$, translating it to the right by 1 unit and down by 3 units, before finally scaling it vertically by the factor -2 .

(These transformations can be carried out in any order, except that the vertical translation must be done before the vertical scaling.)

These three transformations, in the order given above, move the corner of the graph from $(0, 0)$ to $(1, 0)$, then to $(1, -3)$ and finally to $(1, 6)$.

The y -intercept is

$$f(0) = -2(|0 - 1| - 3) = -2(1 - 3) = 4.$$

The x -intercepts are the solutions of the equation $f(x) = 0$. This gives

$$-2(|x - 1| - 3) = 0$$

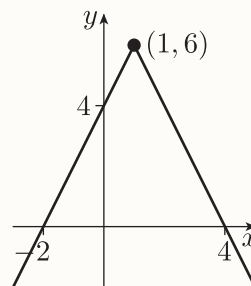
$$|x - 1| - 3 = 0$$

$$|x - 1| = 3$$

$$x - 1 = 3 \quad \text{or} \quad x - 1 = -3$$

$$x = 4 \quad \text{or} \quad x = -2.$$

So the x -intercepts are -2 and 4 .



Solution to Exercise 22

- (a) First we express the rule of f in completed-square form:

$$\begin{aligned} f(x) &= 3x^2 + 18x + 15 \\ &= 3(x^2 + 6x) + 15 \\ &= 3((x + 3)^2 - 9) + 15 \\ &= 3(x + 3)^2 - 27 + 15 \\ &= 3(x + 3)^2 - 12. \end{aligned}$$

So the graph of f can be obtained from the graph of $y = x^2$ by performing the following transformations, in order:

- a vertical scaling by a factor of 3
- a translation to the left by 3 units
- a translation down by 12 units.

(The order of the vertical scaling and the translation to the left can be reversed.)

It follows that the vertex of the graph of f is $(-3, -12)$.

The y -intercept is

$$f(0) = 15.$$

The x -intercepts are the solutions of the equation

$$3x^2 + 18x + 15 = 0,$$

which is equivalent to

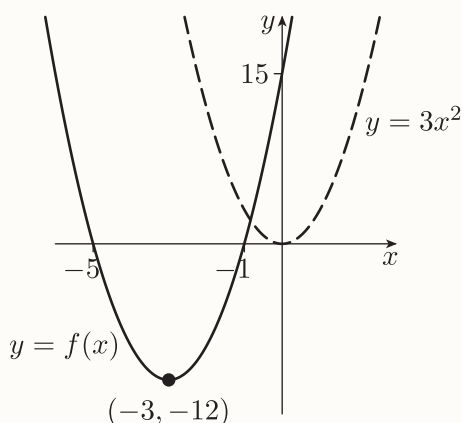
$$x^2 + 6x + 5 = 0;$$

that is,

$$(x + 5)(x + 1) = 0.$$

So the x -intercepts are $x = -5$ and $x = -1$.

The graph of f is the unbroken curve in the following diagram.



- (b) First we express the rule of f in completed-square form:

$$\begin{aligned} f(x) &= -3x^2 + 2x - \frac{7}{3} \\ &= -3\left(x^2 - \frac{2}{3}x\right) - \frac{7}{3} \\ &= -3\left(\left(x - \frac{1}{3}\right)^2 - \frac{1}{9}\right) - \frac{7}{3} \\ &= -3\left(x - \frac{1}{3}\right)^2 + \frac{1}{3} - \frac{7}{3} \\ &= -3\left(x - \frac{1}{3}\right)^2 - 2. \end{aligned}$$

So the graph of f can be obtained from the graph of $y = x^2$ by performing the following transformations, in order:

- a vertical scaling by a factor of -3
- a translation to the right by $\frac{1}{3}$ of a unit
- a translation down by 2 units.

(The order of the vertical scaling and the translation to the right can be reversed.)

It follows that the vertex of the graph of f is $(\frac{1}{3}, -2)$.

The y -intercept is

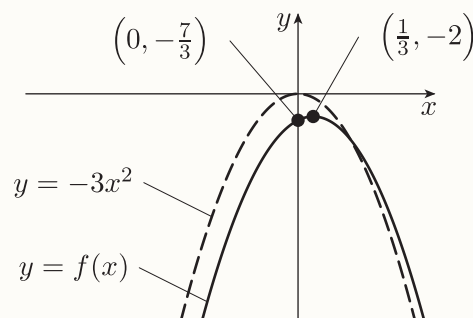
$$f(0) = -\frac{7}{3}.$$

The graph has no x -intercepts, because

$$f(x) = -3\left(x - \frac{1}{3}\right)^2 - 2 \leq -2$$

for all x .

The graph of f is the unbroken curve in the following diagram.



Solution to Exercise 23

- (a) The sum of
- f
- and
- g
- has rule

$$h(x) = (x + 3)^3 + |x| + 11$$

and domain $\mathbb{R}$.

- (b) The difference of
- f
- and
- g
- , with
- g
- subtracted from
- f
- , has rule

$$h(x) = (x + 3)^3 - |x| + 5$$

and domain $\mathbb{R}$.

- (c) The quotient of
- f
- and
- g
- , with
- f
- on the numerator, has rule

$$h(x) = \frac{(x + 3)^3 + 8}{|x| + 3}$$

and domain $\mathbb{R}$.

- (d) The other quotient of
- f
- and
- g
- has rule

$$h(x) = \frac{|x| + 3}{(x + 3)^3 + 8}$$

and domain $(-\infty, -5) \cup (-5, \infty)$.**Solution to Exercise 24**

(a) $(g \circ h)(x) = \sqrt{x - 1}$

(b) $(h \circ g)(x) = \sqrt{x} - 1$

(c) $(f \circ g)(x) = (\sqrt{x})^3 = x^{3/2}$

(d) $(g \circ f)(x) = \sqrt{x^3} = x^{3/2}$

(e) $(f \circ g \circ h)(x) = (\sqrt{x - 1})^3 = (x - 1)^{3/2}$

Solution to Exercise 25

(a) $g \circ f$

(b) $f \circ g$

(c) $h \circ h$

Solution to Exercise 26

- (a) This function is one-to-one.

Its graph is the graph of $y = x^3$ translated by 1 unit up. Since any horizontal line crosses the graph of $y = x^3$ exactly once, the same is true of the graph of the function here. So it is one-to-one.

- (b) This function is not one-to-one. For example,
- $f(0) = f(2) = 1$
- .

- (c) This function is not one-to-one. For example,
- $f(0) = f(1) = 0$
- .

- (d) This function is not one-to-one. Its graph is a parabola with a vertical axis of symmetry, so it is possible to draw a

horizontal line that crosses the graph twice.

Solution to Exercise 27

- (a) The equation
- $f(x) = y$
- gives

$$\frac{(1 - x)}{3} = y$$

$$1 - x = 3y$$

$$x = 1 - 3y.$$

So the rule of f^{-1} is

$$f^{-1}(y) = 1 - 3y;$$

that is,

$$f^{-1}(x) = 1 - 3x.$$

The domain of f^{-1} is the image set of f , which is $\mathbb{R}$. So the inverse function of f is

$$f^{-1}(x) = 1 - 3x.$$

- (b) The equation
- $f(x) = y$
- gives

$$5 - \frac{1}{x} = y$$

$$5 - y = \frac{1}{x}$$

$$x = \frac{1}{5 - y}.$$

So the rule of f^{-1} is

$$f^{-1}(y) = \frac{1}{5 - y};$$

that is,

$$f^{-1}(x) = \frac{1}{5 - x}.$$

The domain of f^{-1} is the image set of f , which is $(-\infty, 5) \cup (5, \infty)$. This is the largest set of real numbers for which the rule of f^{-1} is applicable. So the inverse function of f is

$$f^{-1}(x) = \frac{1}{5 - x}.$$

- (c) The equation
- $f(x) = y$
- gives

$$\frac{1}{5 - x} = y$$

$$\frac{1}{y} = 5 - x$$

$$x = 5 - \frac{1}{y}.$$

So the rule of f^{-1} is

$$f^{-1}(y) = 5 - \frac{1}{y};$$

that is,

$$f^{-1}(x) = 5 - \frac{1}{x}.$$

The domain of f^{-1} is the image set of f , which is $(-\infty, 0) \cup (0, \infty)$. This is the largest set of real numbers for which the rule of f^{-1} is applicable. So the inverse function of f is

$$f^{-1}(x) = 5 - \frac{1}{x}.$$

Solution to Exercise 28

(a) Rearranging the equation $f(x) = y$ gives

$$2 - 3x = y$$

$$y + 3x = 2$$

$$3x = 2 - y$$

$$x = \frac{1}{3}(2 - y).$$

Thus the inverse function is

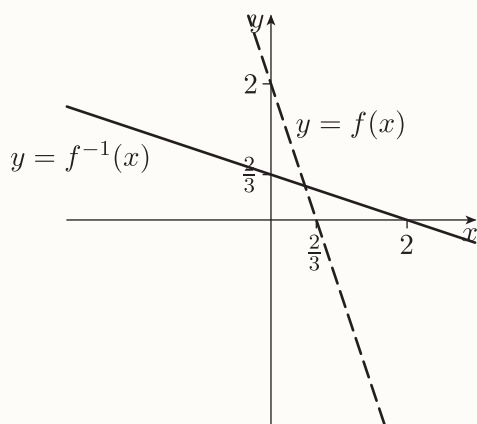
$$f^{-1}(y) = \frac{1}{3}(2 - y),$$

or, in terms of x ,

$$f^{-1}(x) = \frac{1}{3}(2 - x).$$

The domain of f^{-1} is $\mathbb{R}$.

The graph of f^{-1} is the unbroken line in the diagram below. (It is the reflection of the graph of f in the line $y = x$.)



(b) Rearranging the equation $f(x) = y$ gives

$$2x + 2 = y$$

$$2x = y - 2$$

$$x = \frac{1}{2}(y - 2).$$

Thus the inverse function is

$$f^{-1}(y) = \frac{1}{2}(y - 2),$$

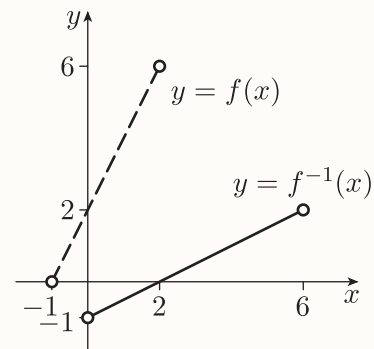
or, in terms of x ,

$$f^{-1}(x) = \frac{1}{2}(x - 2).$$

The domain of f^{-1} is the image set of f , which is

$$(f(-1), f(2)) = (0, 6).$$

The graph of f^{-1} is the unbroken line in the diagram below. (It is the reflection of the graph of f in the line $y = x$.)



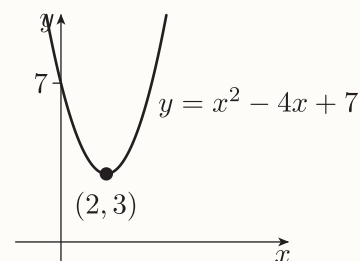
Solution to Exercise 29

(a) Completing the square gives

$$\begin{aligned} f(x) &= x^2 - 4x + 7 \\ &= (x - 2)^2 - 4 + 7 \\ &= (x - 2)^2 + 3. \end{aligned}$$

(b) The graph is a u-shaped parabola with vertex $(2, 3)$. It has y -intercept $f(0) = 7$ and no x -intercepts, since $f(x) = (x - 2)^2 + 3$ is positive for all x .

So the graph is as shown below.



(c) A one-to-one function g that is a restriction of f and has the same image set as f is

$$g(x) = x^2 - 4x + 7 \quad (x \in [2, \infty)).$$

(An alternative choice is

$$g(x) = x^2 - 4x + 7 \quad (x \in (-\infty, 2]).$$

(d) The equation $g(x) = y$ gives

$$(x - 2)^2 + 3 = y$$

$$(x - 2)^2 = y - 3$$

$$x - 2 = \pm \sqrt{y - 3}$$

$$x = 2 \pm \sqrt{y - 3}.$$

Since the domain of g is $[2, \infty)$, each input value x of g is greater than or equal to 2. So

$$x = 2 + \sqrt{y - 3}.$$

Hence the rule of g^{-1} is

$$g^{-1}(y) = 2 + \sqrt{y - 3},$$

that is,

$$g^{-1}(x) = 2 + \sqrt{x - 3}.$$

The domain of g^{-1} is the image set of g , which is $[3, \infty)$. So g^{-1} is given by

$$g^{-1}(x) = 2 + \sqrt{x - 3} \quad (x \in [3, \infty)).$$

(If instead we choose the alternative option for g in part (iii), then we obtain

$$g^{-1}(x) = 2 - \sqrt{x - 3} \quad (x \in [3, \infty)).$$

Solution to Exercise 30

(a) and (c) are approximately equivalent, because

$$1000^x = e^{(\ln 1000)x} \approx e^{6.907755x}.$$

(b) and (e) are approximately equivalent, because

$$0.5^x = e^{(\ln 0.5)x} \approx e^{-0.693147x}.$$

(d) and (f) are approximately equivalent, because

$$2.718282^x \approx e^x.$$

Solution to Exercise 31

(a) $\log_3 1 = 0$

(b) $\log_5\left(\frac{1}{25}\right) = \log_5(5^{-2}) = -2$

(c) $\ln(e^{3.1}) = \log_e(e^{3.1}) = 3.1$

(d) $\log_{10}(\sqrt{1000}) = \log_{10}(10^{3/2}) = \frac{3}{2}$

(e) $e^{\ln 4} = e^{\log_e 4} = 4$

(f) $2^{3 \log_2 5} = (2^{\log_2 5})^3 = 5^3 = 125$

Solution to Exercise 32

(a) $3 = \log_5 x$ gives $5^3 = x$, so $x = 125$.

(b) There is no such value of x , because only positive numbers have logarithms.

(c) $\frac{1}{2} = \log_x 9$ gives $x^{1/2} = 9$, so $x = 9^2 = 81$.

(d) There is no such value of x , because the base of a logarithm must be a positive number not equal to 1.

Solution to Exercise 33

$$\begin{aligned} e^{\ln 2} + \log_5\left(\frac{1}{125}\right) + \ln(4e) - \ln 4 \\ &= 2 + \log_5\left(\frac{1}{5^3}\right) + \ln 4 + \ln e - \ln 4 \\ &= 2 + \log_5(5^{-3}) + \ln e \\ &= 2 + (-3) + 1 \\ &= 0 \end{aligned}$$

Solution to Exercise 34

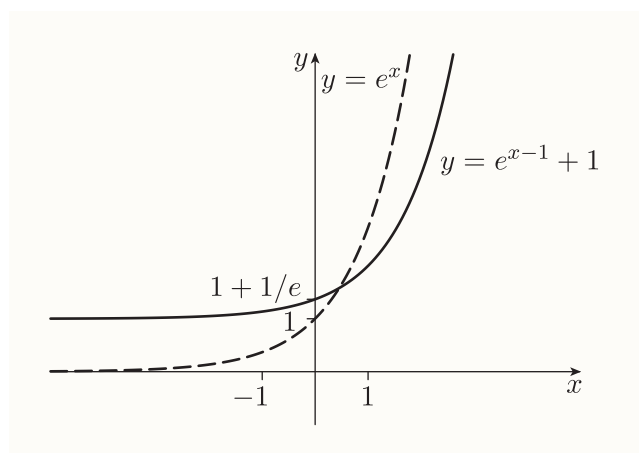
(a) The graph of the function $g(x) = e^{x-1} + 1$ is obtained by translating the graph of the function $f(x) = e^x$ to the right by 1 unit and up by 1 unit.

(Note that since

$$g(x) = e^{x-1} + 1 = e^{-1} \times e^x + 1,$$

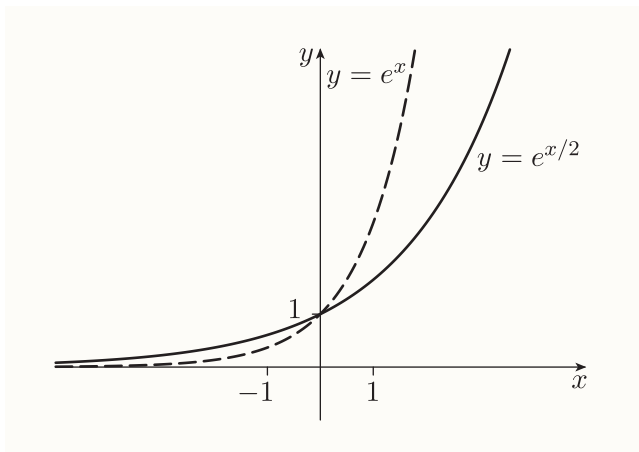
another way to obtain the graph of g is to first scale the graph of f vertically by a factor of $e^{-1} = 1/e$, and then translate it up by 1 unit.)

This gives the following graph.



(b) The graph of the function $h(x) = e^{x/2}$ is obtained by scaling the graph of the function $f(x) = e^x$ horizontally by a factor of 2.

This gives the following graph.



Solution to Exercise 35

- (a) Taking the natural logarithm of both sides gives

$$3^x = 70$$

$$\ln(3^x) = \ln 70$$

$$x \ln 3 = \ln 70$$

$$x = \frac{\ln 70}{\ln 3} = 3.87 \text{ (to 3 s.f.)}.$$

(An alternative method is to take the logarithm to base 3 of each side, which gives

$$x = \log_3 70 = 3.87 \text{ (to 3 s.f.)}.)$$

- (b) Since $81 = 9^2 = 3^4$, you can immediately solve the equation to obtain $x = 4$.

(The approach in part (a) would give

$$x = \frac{\ln 81}{\ln 3} = 4.)$$

Solution to Exercise 36

- (a) $\ln\left(\frac{1}{2}xy^3\right) = \ln \frac{1}{2} + \ln x + \ln(y^3)$
 (by the first law)
 $= \ln(2^{-1}) + \ln x + \ln(y^3)$
 $= -\ln 2 + \ln x + 3 \ln y$
 (by the third law)
 $= \ln x + 3 \ln y - \ln 2$

- (b) $\ln\left(\frac{3x^2 - 12}{e^{2x}}\right)$
 $= \ln(3x^2 - 12) - \ln(e^{2x})$
 (by the second law)
 $= \ln(3(x^2 - 4)) - 2x$
 $= \ln(3(x - 2)(x + 2)) - 2x$
 $= \ln 3 + \ln(x - 2) + \ln(x + 2) - 2x$
 (by the first law)

Solution to Exercise 37

- (a) Let $f(t) = ae^{kt}$, where a and k are constants.

We know that $f(6) = 500$ and $f(8) = 5000$, so

$$ae^{6k} = 500 \quad (1)$$

$$ae^{8k} = 5000. \quad (2)$$

Dividing equation (2) by equation (1) gives

$$\frac{ae^{8k}}{ae^{6k}} = \frac{5000}{500}$$

$$e^{8k-6k} = 10$$

$$e^{2k} = 10$$

$$2k = \ln 10$$

$$k = \frac{1}{2} \ln 10 = 1.1512925465 \dots$$

Equation (1) can be written as

$$a(e^{2k})^3 = 500,$$

and substituting $e^{2k} = 10$ (from the manipulation above) into this equation gives

$$a \times 10^3 = 500$$

$$a = 0.5.$$

Therefore the function f is given by

$$f(t) = 0.5e^{0.5(\ln 10)t} \quad (2 \leq t \leq 12);$$

that is, approximately by

$$f(t) = 0.5e^{1.151t} \quad (2 \leq t \leq 12).$$

- (b) The expected number of bacteria per gram after 12 hours is

$$\begin{aligned} f(12) &= 0.5e^{0.5 \ln 10 \times 12} \\ &= 0.5e^{6 \ln 10} \\ &= 0.5(e^{\ln 10})^6 \\ &= 0.5 \times 10^6 \\ &= 500\,000. \end{aligned}$$

- (c) Every hour the number of bacteria is multiplied by

$$\begin{aligned} e^{(1/2) \ln 10} &= (e^{\ln 10})^{1/2} \\ &= 10^{1/2} \\ &= \sqrt{10} \\ &= 3.162 \text{ (to 4 s.f.)}. \end{aligned}$$

Solution to Exercise 38

- (a) This inequality is true, as it is obtained by multiplying both sides of the original inequality $x > 2$ by the positive number 4.
- (b) This inequality is true, as it is obtained by subtracting 2 from each side of the original inequality $x > 2$.
- (c) This inequality is true, as it is obtained by multiplying both sides of the original inequality by the positive number $\frac{1}{2}$.
- (d) This inequality is false. Multiplying the original inequality by -1 gives $-x < -2$, and it follows that the inequality $-x > -2$ is false.
- (e) This inequality is false. Multiplying the original inequality by $-\frac{1}{2}$ gives $-\frac{x}{2} < -1$, and it follows that the inequality $-\frac{x}{2} > -1$ is false.
- (f) It is not possible to say whether the inequality $ax > 2a$ is true or false, as we do not know whether a is positive or negative.

Solution to Exercise 39

- (a) $5x + 2 < 2x + 11$
 $3x + 2 < 11$
 $3x < 9$
 $x < 3$.
 The solution set is $(-\infty, 3)$.
- (b) $5x - 2 \leq 2x - 11$
 $3x - 2 \leq -11$
 $3x \leq -9$
 $x \leq -3$.
 The solution set is $(-\infty, -3]$.
- (c) $\frac{x-2}{3} > 1 - \frac{x}{2}$
 $x - 2 > 3 - \frac{3x}{2}$
 $\frac{5x}{2} > 5$
 $x > 2$.
 The solution set is $(2, \infty)$.

Solution to Exercise 40

- (a) Rearranging the inequality gives

$$x^2 - x > 6$$

$$x^2 - x - 6 > 0$$

$$(x-3)(x+2) > 0.$$

The graph of $f(x) = x^2 - x + 6$ is u-shaped, with x -intercepts 3 and -2 .

So the solution set of the inequality is $(-\infty, -2) \cup (3, \infty)$.

(Alternatively, instead of considering the graph, you can find the solution set by constructing a table of signs, as below.)

x	$(-\infty, -2)$	-2	$(-2, 3)$	3	$(3, \infty)$
$x - 3$	—	—	—	0	+
$x + 2$	—	0	+	+	+
$(x - 3) \times (x + 2)$	+	0	—	0	+

- (b) Rearranging the inequality gives

$$-2x^2 < 2$$

$$x^2 > -1.$$

As x^2 is always positive, the solution set of the inequality is $\mathbb{R}$.

- (c) Rearranging the inequality gives

$$x^2 \leq 2x - 1$$

$$x^2 - 2x + 1 \leq 0$$

$$(x-1)^2 \leq 0.$$

As the square of a real number is always positive or zero, the only solution occurs when

$$(x-1)^2 = 0;$$

that is, when $x = 1$. So the solution set is $\{1\}$.

Solution to Exercise 41

Rearranging the inequality gives

$$3x + 7 < \frac{20}{3x - 1}$$

$$3x + 7 - \frac{20}{3x - 1} < 0$$

$$\frac{(3x + 7)(3x - 1)}{3x - 1} - \frac{20}{3x - 1} < 0$$

$$\frac{(3x + 7)(3x - 1) - 20}{3x - 1} < 0$$

$$\frac{9x^2 + 18x - 7 - 20}{3x - 1} < 0$$

$$\frac{9x^2 + 18x - 27}{3x - 1} < 0$$

$$\frac{x^2 + 2x - 3}{3x - 1} < 0$$

$$\frac{(x - 1)(x + 3)}{3x - 1} < 0.$$

A factor is equal to 0 when $x = -3$, $x = \frac{1}{3}$ and $x = 1$.

x	$(-\infty, -3)$	-3	$(-3, \frac{1}{3})$	$\frac{1}{3}$	$(\frac{1}{3}, 1)$	1	$(1, \infty)$
$x - 1$	—	—	—	—	—	0	+
$x + 3$	—	0	+	+	+	+	+
$3x - 1$	—	—	—	0	+	+	+
$\frac{(x - 1)(x + 3)}{(3x - 1)}$	—	0	+	*	—	0	+

The table shows that the solution set is $(-\infty, -3) \cup (\frac{1}{3}, 1)$.